Torsion classes realize the Tamari lattice in type A

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Contents

The reference text for this talk is [Tho12].

1 Indecomposables in rep A_n

Let K be a field. Recall that a quiver Q is a directed graph, i.e. the data of a vertex set V, an edge set E, and a collection of maps assigning to each edge α a source and target. Thus we may refer to edges as arrows $\alpha : i \to j$. A quiver representation is the data of a collection of K-vector spaces $\{V_i\}_{i\in V}$ and K-linear maps $\{V_\alpha : V_i \to V_j\}_{\alpha\in E}$ along the arrows $\alpha : i \to j$. We denote by rep Q the abelian category of quiver representations of Q.

We will begin by reviewing some basic vocabulary of representations. Let Y be a representation of Q.

Definition 1.0.1. We say X is a *subrepresentation* of Y if for all i, $X_i \subseteq Y_i$, and the maps $X_\alpha : X_i \to X_j$ are induced by the inclusions $X_i \hookrightarrow Y_i$ as well as $Y_\alpha : Y_i \to Y_j$ for all $\alpha : i \to j$.

Definition 1.0.2. If X is a subrepresentation of Y, the *quotient representation* Y/X is given by $(Y/X)_i = Y_i/X_i$, and maps $(Y/X)_\alpha : (Y/X)_i \to (Y/X)_j$ induced by $X_\alpha : X_i \to X_j$ and $Y_\alpha : Y_i \to Y_j$ for all arrows $\alpha : i \to j$. In particular, we obtain a surjection $Y \twoheadrightarrow Y/X$.

Definition 1.0.3. For X, Y, Z representations of Q, Y is an *extension* of Z by X if Y admits a subrepresentation $W \cong X$, and $Y/W \cong Z$.

We call the extension Y of Z by X *trivial* if $Y \cong X \oplus Z$. We also introduce a notion of a *pullback represetation.*

Lemma 1.0.4. *Let Y be an extension of Z by X. For $h : Z' \twoheadrightarrow Z$ a surjection, there exists a representation Y' an extension of Z' by X with a surjection $Y' \twoheadrightarrow Y$.*

We call Y' the *pullback of Y along h.*

$$\begin{array}{ccc} Y' & \twoheadrightarrow & Z' \\ \downarrow & & \downarrow h \\ Y & \twoheadrightarrow & Z \end{array}$$

Today's protagonist will be type A quivers. These furnish a special family of quivers because their underlying unoriented graphs are the type A Dynkin diagrams, which encode the root system of $\mathfrak{sl}_n$. Consequently, these quivers admit a finite number of indecomposable representations up to isomorphism, these being in one-to-one correspondence with the positive roots of $\mathfrak{sl}_n$.

Definition 1.0.5. Let A_n denote the quiver with vertices $1, \ldots, n$ and arrows $\alpha_i : i \to i+1$, $1 \le i \le n-1$.

$$A_n : \quad \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \cdots \longrightarrow \underset{n}{\bullet}$$

Definition 1.0.6. For $1 \le i \le j \le n$, let E^{ij} be the representation given by 1-dimensional vector spaces at all vertices $i \le p \le j$ with identity maps between them, and zero maps and spaces elsewhere.

Example 1.0.7.

$$\begin{array}{lccccccccc} A_5: & \underset{1}{\bullet} & \longrightarrow & \underset{2}{\bullet} & \longrightarrow & \underset{3}{\bullet} & \longrightarrow & \underset{4}{\bullet} & \longrightarrow & \underset{5}{\bullet} \\ E^{35}: & 0 & \longrightarrow & 0 & \longrightarrow & K & \xrightarrow{\text{id}} & K & \xrightarrow{\text{id}} & K \\ E^{23}: & 0 & \longrightarrow & K & \xrightarrow{\text{id}} & K & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

Proposition 1.0.8. *The representation E^{ij} are indecomposable. Any indecomposable representation of A_n is isomorphic to some E^{ij}.*

Proof. Suppose $E^{ij} \cong X \oplus Y$. Then for $i \le k \le j$, either X_k or Y_k is zero. For $i \le k \le j-1$, if $X_k \ne 0$, $Y_{k+1} \ne 0$ or $Y_k \ne 0$, $X_{k+1} \ne 0$, then along $\alpha : k \to k+1$ the map $(E^{ij})_\alpha = X_\alpha \oplus Y_\alpha = 0$, violating that $(E^{ij})_\alpha$ is the identity. It follows that X or Y is the zero representation, and so E^{ij} is indecomposable.

Let V be an indecomposable representation of A_n, and denote the maps $p_k : V_k \to V_{k+1}$. Let i be minimal such that $V_i \ne 0$, and pick $t \in V_i$ nonzero. Let T be the subrepresentation of V generated by t, with j maximal such that $p_{j-1} \circ \cdots \circ p_i(t) \ne 0$:

$$\begin{array}{lccccccccccc} T: & \cdots & \longrightarrow & T_i & \xrightarrow{p_i|_{T_i}} & \cdots & \xrightarrow{p_{j-1}|_{T_{j-1}}} & T_j & \xrightarrow{p_j|_{T_j}} & 0 & \longrightarrow & \cdots \\ & & & \downarrow & & & & \downarrow & & \downarrow & & \\ V: & \cdots & \longrightarrow & V_i & \xrightarrow{p_i} & \cdots & \xrightarrow{p_{j-1}} & V_j & \xrightarrow{p_j} & V_{j+1} & \longrightarrow & \cdots \end{array}$$

Now, we can define splitting maps $s_k : V_k \to T_k$ inductively (start with s_j the projection $V_j \to T_j$, then for $i \le k \le j-1$, define $s_k := (p_i|_{T_i})^{-1} \circ s_{k+1} \circ p_k$, noting that $p_i|_{T_i}$ is an isomorphism by our choices of j). Letting $s_k = 0$ for all $1 \le k < i$ and $j < k \le n$, one can check that the s_k furnish a morphism $s : V \to T$, and so $V \cong T \oplus (V/T)$. Since V is assumed to be indecomposable, and $T \cong E^{ij} \ne 0$, we deduce $V \cong E^{ij}$.

□

Remark 1.0.9. There are $\binom{n+1}{2}$ indecomposable representations of A_n.

Proposition 1.0.10. *The dimension of the space of morphisms from E^{ij} to $E^{k\ell}$ is*

$$\begin{cases} 1 & \text{if } k \le i \le \ell \le j, \\ 0 & \text{else.} \end{cases}$$

Proof. We will only provide a heuristic here. Let $f : E^{ij} \to E^{k\ell}$ be a morphism of representations. Note that when the condition $k \le i \le \ell \le j$ is not satisfied, we find one of the following two shapes in the commutative diagram for f:

$$
\begin{array}{c}
E^{ij} \\
{\scriptstyle f}\big\downarrow \\
E^{k\ell}
\end{array}
\qquad
\begin{array}{ccc}
K & \xrightarrow{\cong} & K \\
\big\downarrow & & \big\downarrow{\scriptstyle \text{nonzero}} \\
0 & \longrightarrow & K
\end{array}
\qquad
\begin{array}{ccc}
K & \longrightarrow & 0 \\
{\scriptstyle \text{nonzero}}\big\downarrow & & \big\downarrow \\
K & \xrightarrow{\cong} & K
\end{array}
$$

In either case, commutativity fails, since factoring through a zero vector space forces a zero composition. □

Proposition 1.0.11. *Nontrivial extensions of E^{ij} by $E^{k\ell}$ exist only when*

$$i+1 \le k \le j+1 \le \ell.$$

When this holds, all nontrivial extensions are isomorphic to $E^{i\ell} \oplus E^{kj}$. (If $k = j+1$, $E^{kj} = 0$.)

Proof. Let Y be a nontrivial extension of E^{ij} by $E^{k\ell}$, and $t \in Y_i$ an element whose image in $(E^{ij})_i$ under the surjection $h : Y \twoheadrightarrow E^{ij}$ is nonzero. Denote T the subrepresentation generated by t; in particular, $T_k \neq 0$ for $i \le k \le j$. If $T_{j+1} = 0$, then $T \cong E^{ij}$, and thus h splits the inclusion $T \hookrightarrow Y$ meaning $Y \cong E^{ij} \oplus E^{k\ell}$. Since Y is nontrivial, it must hold, then, that $T_{j+1} \neq 0$, meaning that the generator of T_{j+1} includes into the subrepresentation of Y isomorophic to $E^{k\ell}$, since its image under h in $(E^{ij})_{j+1} = 0$ is necessarily zero. Consequently, $k \le j+1 \le \ell$ to ensure $(E^{k\ell})_{j+1} \neq 0$.

Now, we eliminate the case $i \ge k$. Denote by v the image of t in Y_{j+1}, which must include into $(E^{k\ell})_{j+1}$ since $(E^{ij})_{j+1} = 0$. Then we can pullback v via the identity maps to an element $x \in (E^{k\ell})_i \neq 0$ (since $i \ge k$). Replacing T by the subrepresentation T' generated by $t - x$, we deduce $T'_{j+1} = 0$ (while $T'_k \neq 0$ for $i \le k \le j$ since t was chosen with nontrivial image in $(E^{ij})_i$) and so we again obtain a splitting and the conclusion that Y is trivial.

When $i+1 \le k \le j+1 \le \ell$, pick $t \in Y_i$ nonzero. If its image in Y_{j+1} is zero, then the subrepresentation T generated by T is isomorphic to E^{ij}, and so again h splits the inclusion $T \hookrightarrow Y$ and Y is trivial. If its image in Y_{j+1} is nonzero, it must include into $(E^{k\ell})_{j+1}$, which implies $T \cong E^{i\ell}$. As in the proof of Proposition 1.0.8, we can construct a splitting $s : Y \to T$ and discover that $Y \cong E^{i\ell} \oplus E^{kj}$ as desired. □

Example 1.0.12. The indecomposable representations of A_2 are E^{11}, E^{12}, and E^{22}. We have $E^{12}/E^{22} \cong E^{11}$. The indecomposable representations of A_2 are $E^{11}, E^{12}, E^{13}, E^{22}, E^{23}$, and E^{33}. For instance, $E^{13} \oplus E^{22}/E^{23} \cong E^{12}$

The indecomposable representations of E^{ij} furnish the building blocks for the additive subcategories of rep A_n. Their interactions as characterized by the previous propositions will inform the composition and structure of quotient- and extension-closed additive subcategories.

2 Quotient-closed subcategories of rep A_n

We are aiming to classify the *torsion classes* of rep A_n. They are defined as follows.

Definition 2.0.1. A *torsion class* $\mathcal{T}$ in rep A_n is a full additive subcategory closed under

(1) quotients: $Y \in \mathcal{T}$ and $Y \twoheadrightarrow Z \implies Z \in \mathcal{T}$,

(2) extensions: $X, Z \in \mathcal{T}$ and Y and extension of Z by $X \implies Y \in \mathcal{T}$.

We will begin by classifying a slightly larger collection of subcategories, in which we will discover the torsion classes.

Let $\mathbf{M} = \{(a_1, \dots, a_n) : 0 \le a_i \le n+1-i\}$, and define

$$\mathcal{F}_a = \{(i,j) : i \le j \le i + a_i\}.$$

Then let C_a denote the full subcategory consisting of all direct sums of indecomposable representations E^{ij}, $(i,j) \in \mathcal{F}_a$.

Proposition 2.0.2. *The quotient-closed subcategories of rep A_n are exactly C_a, $a \in \boldsymbol{M}$.*

Proof. Quotient-closed subcategories containing E^{ij} contain $E^{ii}, E^{i(i+1)}, \dots, E^{i(j-1)}$ since surjections $E^{ij} \twoheadrightarrow E^{i(j-k)}$ for $1 \le k \le j-i$. If $X \in C_a$, and $X \twoheadrightarrow Y \notin C_a$, then Y must contain a summand E^{ij}, $(i,j) \notin \mathcal{F}_a$. Composing with the projection onto this factor, we obtain $X \twoheadrightarrow E^{ij}$, implying that X contains a summand E^{ik}, $k \ge j$. But $X \in C_a$ implies that $(i,k) \in \mathcal{F}_a$, contradicting that $\mathcal{F}_a$ is “downward closed”. □

Endow $\mathbf{M}$ with the partial order inherited from the Cartesian product: $a \le b$ iff $a_i \le b_i$ for all $i = 1, \dots, n$. Then, one can show that

$$C_a \subset C_b \iff a \le b.$$

This equips the collection of quotient-closed subcategories with a partial order.

Question 2.0.3. Where are the torsion classes of rep A_n? Are there combinatorial criteria we can impose on $a = (a_1, \dots, a_n)$?

3 Torsion classes of rep A_n

Definition 3.0.1. We call $a = (a_1, \dots, a_n)$ a *bracket vector* if, for all $1 \le i \le n$ and $j \le a_i$, we have $j + a_{i+j} \le a_i$.

There is a bijection

$$\left\{\begin{matrix}\text{Bracket vectors}\\ a=(a_1,\ldots,a_n)\end{matrix}\right\} \xleftrightarrow{\sim} \left\{\begin{matrix}\text{Bracket strings}\\ \text{of length } 2n+2\end{matrix}\right\}$$

which can be described as follows: given a string of brackets of length $2n+2$, let a_i be the number of open parentheses strictly between the ith open parenthesis and its corresponding closed parenthesis, $1 \le i \le n+1$. Then $a = (a_1, \ldots, a_n)$ defines a bracket vector (we omit a_{n+1} since it is always necessarily zero). For instance, when $n = 2$ one can check that $()(()) \mapsto (0,1)$, and $(()()) \mapsto (2,0)$.

Theorem 3.0.2. *The torsion classes of rep A_n are exactly C_a for $a = (a_1, \ldots, a_n)$ a bracket vector. Ordered by inclusion, they form a poset isomorphic to the Tamari lattice T_n.*

Remark 3.0.3.

$$|\{C_a : a \in \mathbf{M}\}| = |\mathbf{M}| = (n+1)!$$

$$|\{C_a : a \in \mathbf{M} \text{ a bracket vector}\}| = \frac{1}{n+1}\binom{2n}{n}.$$

What is the Tamari lattice? It was introduced in 1957 by Dov Tamari in his study of parenthesizing strings of n letters, where two parenthesizations may be related by the associativity law:

$$((xy)z) \longrightarrow (x(yz)).$$

Today, the Tamari lattice T_n is known to encode many more combinatorial objects, such as

- triangulations of the $(n+2)$-gon,
- in-ordered binary trees,
- bracket strings of length $2n+2$,
- length-n bracket vectors.

Note that the Tamari lattice is the 1-skeleton of the associahedron.

Lemma 3.0.4. *Let $a = (a_1, \ldots, a_n)$ be a bracket vector. If $X \in C_a$ and $Z \in D_a$, then any extension of Z by X is trivial.*

Proof. Reduce to the case $Z \cong E^{i(i+a_i-1)}$. Let Y be an extension of Z by X, and pick $t \in Y_i$ map to a generator of Z_i. Denote T the subrepresentation generated by t. If the image of t in Y_{i+a_i} is zero, then $T \cong Z$ and so the projection $Y \twoheadrightarrow Z$ splits the inclusion $T \hookrightarrow Y$, i.e. Y is the trivial extension.

So let the image of t in Y_{i+a_i} be $v \neq 0$, i.e. it includes into X_{i+a_i} since $Z_{i+a_i} = 0$. As in the proof of Proposition 1.0.11, this means we can pullback to an element x in X_i whose image in X_{i+a_i} is v. Replacing T by the subrepresentation T' generated by $t-x$, we deduce $T'_{i+a_i}=0$ (while $T'_k\neq 0$ for $i\leq k\leq i+a_i$ since t was chosen with nontrivial image in Z_k) and so we again obtain a splitting and the conclusion that Y is trivial. □

Proof of Theorem 3.0.2. (C_a torsion class $\implies a$ bracket vector): Suppose a is not a bracket vector. Then $\exists(i,j), 1\leq i\leq n, j\geq a_i$ with $j+a_i>a_i$. It suffices to show C_a is *not* closed under extensions. We have $E^{i(i+a_i-1)}, E^{i+j(i+j+a_{i+j}-1)}\in C_a$. Since $j+a_{i+j}>a_i$, we have $i+j+a_{i+j}-1\geq(i+a_i-1)+1$ and so by Proposition 1.0.11, $E^{i(i+j+a_{i+j}-1)}\oplus E^{i+j(i+a_i-1)}$ is a nontrivial extension, except $E^{i(i+j+a_{i+j}-1)}\notin C_a$ because $i+j+a_{i+j}-1\not\leq i+a_i-1$. So C_a is not a torsion class.

(a bracket vector $\implies C_a$ torsion class): Let a be a bracket vector. We've established that C_a is quotient-closed, so it suffices to check closed under extensions. Let Y be an extension of Z by X. Then choose $Z'\in D_a$ such that $Z'\twoheadrightarrow Z$. Denote Y' the pullback along $Z'\twoheadrightarrow Z$. By Lemma 3.0.4, Y' must be trivial, and $Y'\in C_a\implies Y\in C_a$ by quotient-closed.

□

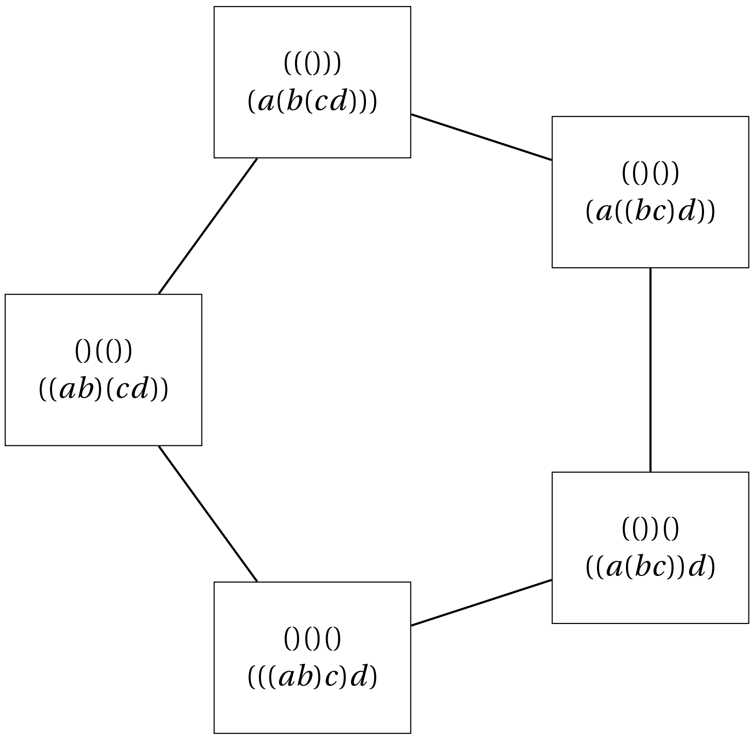

Figure 1: The Tamari Lattice T_3.

References

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